

THERMODYNAMIC BETHE ANSATZ FOR $\mathcal{G}_k \otimes \mathcal{G}_l / \mathcal{G}_{k+l}$ COSET MODELS PERTURBED BY THEIR $\phi_{1,1,Adj}$ OPERATOR

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Abstract

We propose a Thermodynamic Bethe Ansatz (TBA) for $\mathcal{G}_k \otimes \mathcal{G}_l / \mathcal{G}_{k+l}$ conformal coset models (\mathcal{G} any simply-laced Lie algebra) perturbed by their operator $\phi_{1,1,Adj}$. An interesting adjacency structure appears and can be depicted in a sort of “product” of Dynkin diagrams of \mathcal{G} and A_{k+l-1} . UV and IR limits are computed and reproduce the expected values for the central charges. For $k \rightarrow \infty$, l fixed we obtain the TBA of the \mathcal{G}_l WZW model perturbed by $J_a \bar{J}_a$, and for $k, l \rightarrow \infty$, $k - l$ fixed, that of Principal Chiral model with WZ term at level $k - l$.

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Many two dimensional Quantum Field Theories obtained by deformation of a Conformal Field Theory (CFT) by one of its relevant operators show an infinite number of conserved currents that guarantee integrability. There have been various proposals to identify the factorizable S-matrix of such a theory, by use of local [1] or non-local [2, 3] conserved currents.

Recently Al.Zamolodchikov [4] proposed the use of *Thermodynamic Bethe Ansatz* (TBA) techniques to analyze the ultraviolet (UV) and infrared (IR) behaviours of a theory with given S-matrix and to follow the RG flows of various operators. Comparing the results with those expected from perturbed CFT, one can provide a non-trivial check for the S-matrices conjectured via the methods of [1, 2, 3]. For the case of *diagonal* S-matrices, the TBA system of coupled non-linear integral equations determining the thermodynamics of the relativistic scattering theory are relatively simple to obtain and for large sets of such S-matrices they have been explored [8], the central charges, the scaling dimensions of the perturbing operator and some other data of the UV theory being reproduced as expected.

TBA equations are much more difficult to derive in the case of *non-diagonal* S-matrices, where in principle one needs to diagonalize the so called *color transfer matrix*, via higher level Bethe ansatz, a formidable problem in many cases. Nevertheless, TBA equations have been proposed for minimal models perturbed by its least relevant operator ϕ_{13} , for negative [11, 5] values of the coupling. These theories have an S-matrix [3] describing the scattering of kinks, all of the same mass M , separating vacua of different colors. Moreover, one can modify this set of TBA equations, in order to describe the same model with positive coupling, which consists in a massless theory flowing to the nearest lower model in the stair of minimal CFT's [6]. Finally, another modification of the same TBA allows to describe flows between generalizations of minimal models, namely coset models of the kind $SU(2)_k \otimes SU(2)_l / SU(2)_{k+l}$ [7] (that we shall denote for short $[A_1]^{k,l}$ in the following). In [5, 6, 7] the TBA equations are first conjectured on reasonable physical grounds, then from them one can extract non trivial information that can be com-

pared with conformal perturbation theory to confirm that the proposed TBA does indeed describe the expected CFT, deformed by the chosen operator. Once a TBA is (out of reasonable doubt) attached to a certain deformed UV CFT, one can use it to follow the Renormalization Group (RG) flow of the theory. In particular, the IR behaviour can be recovered as the TBA equations are exactly solvable in this limit. By use of this approach, Al.Zamolodchikov was able to check explicitly some previously conjectured features of two-dimensional RG flows, as e.g. the fact that the $[A_1]^{k,l}$ models perturbed by their least relevant operator for positive coupling flow to the $[A_1]^{k-l,l}$ ones. Even more surprisingly, some completely new RG flows connecting non trivial UV and IR points have been discovered, e.g. V.Fateev and Al.Zamolodchikov give evidence of new flows from Z_{k+1} parafermionic models to $[A_1]^{k,1}$ minimal models [10].

An interesting question is that of trying to generalize these results to the large class of $\mathcal{G}_k \otimes \mathcal{G}_l / \mathcal{G}_{k+l}$ coset CFT's, i.e. to $[\mathcal{G}]^{k,l}$ in the previous notation, where \mathcal{G} is *any* Lie algebra [9]. Unfortunately, for \mathcal{G} non-simply laced, the situation on the knowledge of S-matrices is still confused, therefore in the following we shall restrict to the class of simply-laced Lie algebras A, D, E .

Perturbing these models by their scalar relevant operator $\phi_{1,1,Adj}$,¹ in the present paper we consider the models $[\mathcal{G}]_{\pm}^{k,l}$ defined by the following action

$$\mathcal{A}_{[\mathcal{G}]_{\pm}^{k,l}} = \mathcal{A}_{[\mathcal{G}]^{k,l}} \pm \lambda \int d^2z \phi_{1,1,Adj} \quad , \quad \lambda \geq 0 \quad (1)$$

In analogy with the known $\mathcal{G} = A_1$ case, we expect that the model $[\mathcal{G}]_{-}^{k,l}$ will be dominated by massive kinks connecting the different vacua of the theory, while the model $[\mathcal{G}]_{+}^{k,l}$ will define a massless theory flowing to an IR limit identified with the $[\mathcal{G}]^{k-l,l}$ CFT.

As a startpoint of our analysis, we consider the models $[\mathcal{G}]_{-}^{k,1}$. The integrability of these models has been discussed in [13] and kink S-matrices have been proposed

¹This operator has conformal dimension $\Delta = \frac{k+l}{k+l+g}$, g being the dual Coxeter number of \mathcal{G} . The 3 indices label \mathcal{G}_k , \mathcal{G}_l and \mathcal{G}_{k+l} integrable representations respectively.

in [14, 24]. They are non-diagonal, with the exception of $k = 1$, which corresponds to the case studied in [8]. When a non-diagonal S-matrix appears, one has to diagonalize the color transfer matrix. This leads to Bethe equations, that can be interpreted as periodicity conditions on a system of fictitious particles carrying no energy, usually called *magnons*, moving and scattering in a “frame lattice” of physical particles. (see e.g. [26] and refs. therein). To find the details of the magnonic structure one should solve the Bethe equations, which is notably a very difficult task. In the case we are examining, the diagonalization of the color transfer matrix is equivalent to that for the transfer matrix of an IRF model [16] at criticality. This diagonalization has been carried out by Bazhanov and Reshetikhin [15], who also studied the thermodynamic of the corresponding one-dimensional model and of its scaling limit. To be precise, in the solution proposed in [15] there are some assumptions: first of all the assumption that the solution organize themselves in the thermodynamic limit as a set of *strings*. The string hypothesis is not always true (see e.g. [18]) but it is quite reasonable here. Moreover, by analogy with known cases, they assume that some of the strings have densities that vanish in the thermodynamic limit. Finally, all the machinery is explicit for A_N algebras only, and is then *conjectured* for D_N and $E_{6,7,8}$. Nevertheless, we shall see that these assumptions lead to reasonable results.

The TBA for our scattering theory that can be deduced from [15] can be presented in many ways. In this paper we write two equivalent forms of it. The first one is perhaps more suitable for practical calculations, while the second shows a high degree of universality (in the spirit of ref. [12]), allowing a deeper physical understanding. Let us begin, then, by giving the TBA equations in the following form, in which the generalization of the already known results for $[A_1]_-^{k,1}$ [11, 5] and $[\mathcal{G}]_{\pm}^{1,1}$ [8] clearly appears:

$$\nu_i^a(\theta) = \varepsilon_i^a(\theta) + \frac{1}{2\pi} \sum_{b=1}^r \left[\phi^{ab} * L_i^b(\theta) - \sum_{j=1}^k (A_k)_{ij} \psi^{ab} * L_j^b(\theta) \right] \quad (2)$$

Here $i = 1, \dots, k$, $a = 1, \dots, r = \text{rank } \mathcal{G}$, $L_i^a(\theta)$ is short for $\log(1 + e^{-\varepsilon_i^a(\theta)})$ and $A * B(\theta)$

denotes the convolution $\int_{-\infty}^{+\infty} A(\theta - \theta')B(\theta')d\theta'$. For the $[\mathcal{G}]_{-}^{k,1}$ theories the energy terms ν_i^a are chosen as $\nu_i^a = \delta_{i,1}M_a \cosh \theta$, where $M_a = M\psi_a^{\mathcal{G}}$ are the masses for the $k = 1$ scattering theory. (\mathcal{G}) stands for the incidence matrix of \mathcal{G} and $\psi_a^{\mathcal{G}}$ are the components of its Perron-Frobenius eigenvector. The pseudoenergies ε_i^a determined as solutions to this system, enter the formula for the free energy, which in turn can be interpreted as the Casimir energy of the system on a periodic strip

$$E(R) = \frac{\pi c(R)}{6R} = - \sum_{a=1}^r \int_{-\infty}^{+\infty} L_i^a(\theta) \nu_i^a(\theta) d\theta \quad (3)$$

From this, the value of the central charge of the UV theory can be easily estimated. The two kernels ϕ^{ab} and ψ^{ab} can be directly derived from [15] and describe the scattering without color flip and with color flip respectively. It is not surprising that ϕ^{ab} has the same form as for the “colorless” $k = 1$ theories [8]

$$\phi^{ab}(\theta) = -i \frac{d}{d\theta} \log S^{ab}(\theta) \quad (4)$$

where S^{ab} is the diagonal S-matrix for $[\mathcal{G}]_{\pm}^{1,1}$ theories, which can always be put in the form [8, 19]

$$S^{ab}(\theta) = \prod_{\alpha \in A_{ab}} f_{\alpha - \frac{1}{g}}(\theta) f_{\alpha + \frac{1}{g}}(\theta) \quad , \quad f_{\alpha}(\theta) = \frac{\sinh \frac{1}{2}(\theta + i\alpha\pi)}{\sinh \frac{1}{2}(\theta - i\alpha\pi)} \quad (5)$$

where A_{ab} is a set of rational numbers α with common denominator g (for details see [19]). If color flip is allowed, then one has to take into account the second term too, where a new kernel has to be introduced

$$\psi^{ab}(\theta) = -i \frac{d}{d\theta} \log T^{ab}(\theta) \quad (6)$$

T^{ab} is a fictitious S-matrix describing scattering with change of colors $i \rightarrow i \pm 1$ (according to the admissibility A_k Dynkin diagram appearing in eq.(2)). Its general form is

$$T^{ab}(\theta) = \prod_{\alpha \in A_{ab}} f_{\alpha}(\theta) \quad (7)$$

where the product runs over the *same* set A_{ab} as for S^{ab} . In [21] it has been shown that T^{ab} satisfies the same bootstrap equations as S^{ab} . Introducing

$$P_{\alpha}(\theta) = -i \frac{d}{d\theta} \log f_{\alpha}(\theta) = \frac{-\sin \pi \alpha}{\cosh \theta - \cos \pi \alpha} \quad (8)$$

one can write

$$\phi^{ab} = \sum_{\alpha \in A_{ab}} (P_{\alpha - \frac{1}{g}} + P_{\alpha + \frac{1}{g}}) \quad , \quad \psi^{ab} = \sum_{\alpha \in A_{ab}} P_{\alpha} \quad (9)$$

Consider now the Fourier transform

$$\tilde{P}_{\alpha}(\kappa) = \int_{-\infty}^{+\infty} P_{\alpha}(\theta) e^{i\kappa\theta} d\theta = \begin{cases} \frac{\sinh(1-\alpha)\pi\kappa}{\sinh \pi\kappa} & \text{for } \alpha > 0 \\ 0 & \text{for } \alpha = 0 \end{cases} \quad (10)$$

By use of simple trigonometric identities connecting $\tilde{P}_{\alpha - \frac{1}{g}} + \tilde{P}_{\alpha + \frac{1}{g}}$ to \tilde{P}_{α} one can prove the following identity between Fourier transforms $\tilde{\phi}^{ab}(\kappa)$ and $\tilde{\psi}^{ab}(\kappa)$

$$\frac{1}{2\pi} \tilde{\psi}^{ab}(\kappa) = \tilde{R}_g(\kappa) \left[\delta^{ab} - \frac{1}{2\pi} \tilde{\phi}^{ab}(\kappa) \right] \quad (11)$$

where $\tilde{R}_g(\kappa)$ is the universal (i.e. independent of a, b) function

$$\tilde{R}_g(\kappa) = \frac{1}{2 \cosh \frac{\pi\kappa}{g}} \quad (12)$$

The identity (11) together with another useful identity quoted in [12]

$$\sum_{b=1}^r \left[\delta^{ab} - \frac{1}{2\pi} \tilde{\phi}^{ab} \right] \left[\delta^{bc} - (\mathcal{G})^{bc} \tilde{R} \right] = \delta^{ac} \quad (13)$$

allows to re-express system (2) in a much more universal form. Multiply the Fourier transform of both members of (2) by $\delta^{ad} - (\mathcal{G})^{ad} \tilde{R}$, then sum over a . Use of (11) and (13) results in the following form for TBA equations

$$\nu_i^a = \varepsilon_i^a + \frac{1}{2\pi} \varphi_g * \left\{ \sum_{b=1}^r (\mathcal{G})^{ab} [\nu_i^b - \Lambda_i^b] - \sum_{j=1}^k (A_k)_{ij} L_j^a \right\} \quad (14)$$

which is a direct generalization of the universal form found in [12]. Here Λ_i^a is short for $\log(1 + e^{\varepsilon_i^a})$ and

$$\varphi_g(\theta) = 2\pi R_g(\theta) = \frac{g}{2 \cosh \frac{g\theta}{2}} \quad (15)$$

is exactly the same universal kernel of [12].

By analogy with what done in [12, 5, 6, 7], it is interesting to depict system (14) in a graphical form. Here it is natural to consider a “bidimensional” graph that can

be thought as a “product” of the two graphs encoded in (\mathcal{G}) and (A_k) (see Fig.1). Non zero energy terms are attached to the base ($i = 1$) of the graph. The k replicas of the (\mathcal{G}) graph correspond to the k different magnonic structures responsible of exchange of colors. Colors can not change arbitrarily during scattering, but only according to the (A_k) admissibility diagram. This graphical structure helps to propose sets of TBA equations for $[\mathcal{G}]_+^{k,1}$ too, along the lines of ref. [6]. The simple modification is to take now $\nu_i^a = \delta_{i,1} \frac{M_a}{2} e^\theta + \delta_{i,k} \frac{M_a}{2} e^{-\theta}$. The physical interpretation, however is quite different: the two pieces proportional to e^θ and to $e^{-\theta}$ are the energy terms for massless left and right movers respectively. Again $M_a = M \psi_a^{\mathcal{G}}$, but the parameter M is here better interpreted as a crossover scale [17].

Following then ref. [7] one can generalize even more the TBA to include all the models $[\mathcal{G}]_\pm^{k,l}$ just by taking energy terms as follows

$$\nu_i^a(\theta) = \begin{cases} \delta_{i,l} M_a \cosh \theta & \text{for } [\mathcal{G}]_-^{k,l} \\ \delta_{i,l} \frac{M_a}{2} e^\theta + \delta_{i,k} \frac{M_a}{2} e^{-\theta} & \text{for } [\mathcal{G}]_+^{k,l} \end{cases} \quad (16)$$

The graphs on which to encode the TBA are now $(\mathcal{G}) \times (A_{k+l-1})$. The most general TBA set of equation for the $[\mathcal{G}]_\pm^{k,l}$ models then reads as

$$\nu_i^a = \varepsilon_i^a + \frac{1}{2\pi} \varphi_g * \left\{ \sum_{b=1}^r (\mathcal{G})^{ab} [\nu_i^b - \Lambda_i^b] - \sum_{j=1}^{k+l-1} (A_{k+l-1})_{ij} L_j^a \right\} \quad (17)$$

To compute the free energy in the UV-limit, it is better to return to the form (2) of TBA equations, where of course now ν_i^a are given in the more general form (16) and the sum over j runs from 1 to $k + l - 1$. The UV behavior ($R \rightarrow 0$) of pseudoenergies is dominated by the solutions of the shifted “kink” system [4, 8, 6]

$$\psi_a^{\mathcal{G}} e^\theta \delta_{i,l} = \hat{\varepsilon}_i^a + \frac{1}{2\pi} \sum_{b=1}^r \sum_{j=1}^{k+l-1} [\phi^{ab} \delta_{ij} - \psi^{ab} (A_{k+l-1})_{ij}] * \hat{L}_j^k \quad (18)$$

where $\hat{\varepsilon}_i^a$ are the shifted pseudoenergies such that $\hat{\varepsilon}_i^a(0) = \varepsilon_i^a(-\infty)$. This implies, after some standard manipulations, the following expression for c_{UV} in terms of

Rogers Dilogarithm function $\mathcal{L}(x) = -\frac{1}{2} \int_0^x dt \left(\frac{\log(1-t)}{t} + \frac{\log t}{1-t} \right)$

$$c_{UV} = \lim_{R \rightarrow 0} \frac{6RE(R)}{\pi} = \frac{6}{\pi^2} \sum_{a=1}^r \left\{ \sum_{i=1}^{k+l-1} \mathcal{L} \left(\frac{Y_i^a(-\infty)}{1 + Y_i^a(-\infty)} \right) - \sum_{i=1}^{l-1} \mathcal{L} \left(\frac{Y_i^a(+\infty)}{1 + Y_i^a(+\infty)} \right) - \sum_{i=l+1}^{k+l-1} \mathcal{L} \left(\frac{Y_i^a(+\infty)}{1 + Y_i^a(+\infty)} \right) \right\} \quad (19)$$

where $Y_i^a(\theta) = e^{-\varepsilon_i^a(\theta)}$. $Y_i^a(\pm\infty)$ are determined by the following algebraic nonlinear system of equations (use of (11) and (13) for $\kappa = 0$ has been invoked here)

$$y_i^a = \prod_{b=1}^r \left[(1 + y_i^a)^{(1-2K_{\mathcal{G}})^{ab}} \prod_{j=1}^Q (1 + y_j^b)^{(A_{\mathcal{Q}})_{ij} K_{\mathcal{G}}^{ab}} \right] \quad (20)$$

where $Q = k + l - 1$ if $y_i^a = Y_i^a(-\infty)$ and $Q = l - 1$ or $Q = k - 1$ if $y_i^a = Y_i^a(+\infty)$ in the second and third sum of eq.(19) respectively, and $K_{\mathcal{G}}$ is the inverse of the Cartan matrix of \mathcal{G} .

The real solutions of this system satisfy Dilogarithm sum rules

$$\sum_{b=1}^r \sum_{i=1}^Q \mathcal{L} \left(\frac{1}{1 + y_i^a} \right) = \frac{rg(Q-1)}{Q+g} \quad (21)$$

Some of them are proven [15]. The others can be checked numerically with high precision. Together with the identity [20]

$$\mathcal{L} \left(\frac{x}{1+x} \right) = \frac{\pi^2}{6} - \mathcal{L} \left(\frac{1}{1+x} \right) \quad (22)$$

they arrange the sum (19) into

$$c_{UV} = c(\mathcal{G}_k) + c(\mathcal{G}_l) - c(\mathcal{G}_{k+l}) \quad (23)$$

where $c(\mathcal{G}_m) = m \dim \mathcal{G} / (m + g)$ is the central charge of the \mathcal{G} -WZW model at level m , i.e. exactly what expected for the $[\mathcal{G}]^{k,l}$ model from GKO construction [22].

The IR-limit is quite different for $[\mathcal{G}]_-^{k,l}$ than for $[\mathcal{G}]_+^{k,l}$. In the first case the theory is dominated by massive kinks. One could compute the free energy in the $R \rightarrow \infty$ limit and show that the expected behaviour predicted by the interkink statistics is reproduced. We leave this aspect for further work. We are more interested here

to give evidence of RG flows $[\mathcal{G}]^{k,l} \rightarrow [\mathcal{G}]^{k-l,l}$ described by the $[\mathcal{G}]_+^{k,l}$ theory. This simply follows by applying the same arguments as in [6, 7] (suitably generalized), to prove that

$$c_{IR} = c(\mathcal{G}_{k-l}) + c(\mathcal{G}_l) - c(\mathcal{G}_k) \quad (24)$$

Notice that for $k = l$ this gives $c_{IR} = 0$, thus signaling the fact that $[\mathcal{G}]_+^{k,k} \equiv [\mathcal{G}]_-^{k,k}$ is indeed massive (see ref. [7]).

Finally we would like to stress that the limit $k \rightarrow \infty$ at fixed l of $[\mathcal{G}]_-^{k,l}$ TBA, gives a TBA based on a semi-infinite tower of \mathcal{G} Dynkin diagrams. In analogy to the comments in [7] this should correspond to the appropriate TBA for the \mathcal{G} WZW model at level l with asymptotically free perturbation $J_a \bar{J}_a$. Indeed the UV central charge is $\frac{l \dim \mathcal{G}}{l+g}$ in this limit. Of course such a limit can not be defined for the $[\mathcal{G}]_+^{k,l}$ TBA, while in this case it is reasonable to take the limit of both $k, l \rightarrow \infty$, while keeping $k - l = n$ fixed. This, again generalizing the ideas of [7], gives an infinite (in both directions) tower of \mathcal{G} Dynkin diagrams that encode a TBA suitable to describe the Principal Chiral Model with Wess-Zumino term, which is known to give $\dim \mathcal{G}$ free bosons at its UV limit, and the \mathcal{G} WZW model at level n at its IR. Indeed, the UV central charge is $c_{UV} = \dim \mathcal{G}$ here, while the IR one amounts to $c_{IR} = \frac{n \dim \mathcal{G}}{n+g}$. It would be interesting to re-obtain these results by generalizing the approach of ref. [17].

To conclude, making use of a diagonalization of the transfer matrix for IRF models by Bazhanov and Reshetikhin, we have proposed here TBA equations for all the $\mathcal{G}_k \otimes \mathcal{G}_l / \mathcal{G}_{k+l}$ coset models perturbed by their $\phi_{1,1,Adj}$ operator, both in massive and massless directions. We studied the UV and IR limits, recovering the expected central charges. One should also compute the next corrections to the UV limit of the free energy, in order to estimate the bulk term (with or without logarithms), and to compare the next regular terms with the informations from conformal perturbation theory. Also, the corrections to the IR behaviour should be compared to the (non renormalizable) perturbation expansion of the IR effective action for massless theories and to the particle cluster expansion in the massive case.

We leave all this work and more details on the derivation of TBA to a forthcoming more extensive paper.

Acknowledgements and Notes added

V.A.Fateev and Al.B.Zamolodchikov are also aware of the same results [23], that they obtained in a very different way combining many theories into a diagonal S-matrix one, for which TBA can be easily obtained. The idea to study this problem was generated during illuminating conversations with both of them. For this and for many other discussions they are greatly acknowledged.

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While this work was towards its end in mid January 1992, a preprint by M.Martins appeared [25], where the particular case of $[A_2]_{\pm}^{k,1}$ is studied in detail, and a TBA is proposed for $[A_N]_{\pm}^{k,1}$. Where they overlap, his results are in perfect agreement with mine.

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Fig.1 - TBA graph for $[A_4]_{\pm}^{k,l}$ theory.

